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Mizoguchi- Takahashi Fixed Point Theorem In ν-Generalized Metric Spaces

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Abstract: Mizoguchi- Takahashi's fixed point theorem (1989) is a real extension of Nadler's fixed point theorem (1969) in metric spaces. In this paper we will prove Mizoguchi- Takahashi's fixed point theorem in the ν- generalized metric space. Moreover, we prove two more theorems which generalized Mizoguchi- Takahashi' theorem to the new setting.

Keywords: ν- generalized metric space; fixed point; Mizoguchi and Takahashi's theorem.

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1. Introduction

A metric on a nonempty set X is a mapping $d : X \times X \to [0, \infty)$ satisfying the following properties:

- $d(x, y) = 0$ if and only if $x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \le d(x, z) + d(z, y)$ (triangle inequality) for any $x, y, z \in X$

If d is a metric on X, then the pair (X, d) is called a metric space.
The theory of metric spaces form a basic environment for a lot of concepts in mathematics such as the fixed point theorems which play an important rules in different branches of mathematical analysis. One of the famous result of fixed point theorems was the well-known Banach Contraction Principle.

Theorem 1. *[1](Banach Contraction Principle)*
Let (X, d) be a complete metric space and $T : X \to X$ be a map satisfies:

$$d(Tx, Ty) \le rd(x, y)$$

for each $x, y \in X$ where $r \in [0, 1)$. Then T has a unique fixed point.

This theorem has been used and extended in many different directions. For examples, we refer the reader to the following papers [2–5], and the references therein. In (1969) Nadler extended it for multivalued mappings. Let $CB(X)$ denote the set of all nonempty, bounded and closed subsets of X. For any $A, B \in CB(X)$ a Hausdorff metric is defined as

$$H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}$$

Theorem 2. *[6](Nadler's fixed point theorem)*
Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a map satisfies:

$$H(Tx, Ty) \leq rd(x, y)$$

for each $x, y \in X$ where $r \in [0, 1)$. Then T has a fixed point.

Many attempts have been done to generalize Nadler's fixed point theorem. Mizoguchi and Takahashi's fixed point theorem is one of these generalizations.

Theorem 3. *[7](Mizoguchi and Takahashi's fixed point theorem) Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a map satisfies:*

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y)$$

for each $x, y \in X$ where $\alpha : [0, \infty) \to [0, 1)$ is a function such that $\limsup_{s \to t^+} \alpha(s) < 1$. Then T has a fixed point.

Remark 1. *The function α in Mizoguchi and Takahashi's fixed point theorem (MT- theorem for short) which satisfies $\limsup_{s \to t^+} \alpha(s) < 1$ is called MT- function.*

Starting with Mizoguchi and Takahashi's paper, many generalizations of their theorem have been established see [17,18]. Recently, Eldred et al [8], claimed that Nadler's fixed point theorem is equivalent to Mizoguchi and Takahashi's fixed point theorem. However, in [9], Suzuki proved that their claim is not true and he shown that Mizoguchi and Takahashi's fixed point theorem is a real generalization of Nadler's fixed point theorem. This is why we are interesting in such theorem.

In another direction, modifying the triangle inequality in the basic definition of metric space leads to a new concept which was introduced by Branciari in (2000).

Definition 1. *[10] Let $d : X \times X \to [0, \infty)$ and let $\nu \in N$. Then (X, d) is said to be a ν- generalized metric space if the following satisfy:*

(M1) $d(x, y) = 0$ *if and only if* $x = y$;
(M2) $d(x, y) = d(y, x)$;
(M3) $d(x, y) \leq d(x, u_1) + d(u_1, u_2) + ... + d(u_\nu, y)$ *for any $x, u_1, u_2, ...u_\nu, y \in X$ such that $x, u_1, u_2, ...u_\nu, y$ are all different.*

It is not difficult to show that the new space is not the same as the basic one. Moreover, the new space is difficult to deal with because it does not have a compatible topology as Suzuki shown in[11]. Recently, in [12], Suzuki proved Nadler's fixed point theorem in ν- generalized metric space. In this paper, we prove Mizoguchi and Takahashi's fixed point theorem in ν- generalized metric space. In the following section, we recall some basic definitions and results. After that we will prove our main results.

2. Preliminaries

Definition 2. *Let $T : X \to 2^X$ be a multivalued map on X. A point $x \in X$ is said to be a fixed point if $x \in Tx$.*

Definition 3. *[10] Let (X, d) be a ν- generalized metric space. A sequence $\{x_n\}_{n \in N} \in X$ is said to be Cauchy sequence if*

$$\lim_n \sup_{m > n} d(x_m, x_n) = 0$$

Definition 4. *[12] A sequence $\{x_n\}_{n\in N}$ is said to be $(\sum, \neq)$- Cauchy sequence if all x_n's are different and*

$$\sum_{i=1}^{\infty} d(x_i, x_{i+1}) < \infty$$

Definition 5. *[12] Let (X, d) be a ν- generalized metric space. X is a $(\sum, \neq)$- complete if every $(\sum, \neq)$- Cauchy sequence converges.*

Lemma 1. *[12,13]*

- *Let (X, d) be a ν- generalized metric space, and let $\{x_n\}$ be a $(\sum, \neq)$- Cauchy sequence. Then if $\{x_n\}$ converge then it is Cauchy.*
- *Let (X, d) be a ν- generalized metric space, and let $\{x_n\}$ be a Cauchy sequence converges to some $z \in X$. Let $\{y_n\}$ be a sequence in X such that $\lim d(x_n, y_n) = 0$. Then, $\{y_n\}$ converges also to z.*

Lemma 2. *[9] If $\alpha : [0, \infty) \to [0, 1)$ is a MT-function then, for all $t \in [0, \infty)$ there exist $r_t \in [0, 1)$ and $\epsilon_t > 0$ such that $\alpha(s) \le r_t$ for all $s \in [t, t + \epsilon_t)$*

Lemma 3. *[6] Let (X, d) be a metric space. For any $A, B \in CB(X)$ and $\epsilon > 0$, there exist $a \in A$ and $b \in B$ such that $d(a, b) \le H(A, B) + \epsilon$*

3. Main Results

In this section we will state and prove Mizoguchi and Takahashi's fixed point theorem in ν -generalized metric space and some of its generalizations in the space.

Theorem 4. *Let (X, d) be a $(\sum, \neq)$ complete ν- generalized metric space and let $T : X \to CB(X)$ be a multi valued mapping satisfy the following:*

(i) For any $x \in X$, Tx is a nonempty set
(ii) If $\{y_n\} \in Tx$ and $\{y_n\}$ converge to y then $y \in Tx$
(iii) For any $x, y \in X$, $H(Tx, Ty) \le \alpha(d(x, y))d(x, y)$

where α is MT-function. Then T has a fixed point.

Proof. Define $\beta : [0, \infty) \to [0, 1)$ by $\beta(t) = \dfrac{1 + \alpha(t)}{2}$. Then it is not difficult to show that $\alpha(t) < \beta(t)$ for each $t \in [0, \infty)$ and $\lim_{s\to t^+} \sup \beta(s) < 1$. Moreover, for any $x, y \in X$ and $u \in Tx$, there exist $v \in Ty$ such that

$$d(u, v) \le \beta(d(x, y))d(x, y).$$

Putting $u = y$, we obtain the following:

$$d(y, v) \le \beta(d(x, y))d(x, y)$$

Define $f(x) = \inf\{d(x, b) : b \in Tx\}$ and suppose that T does not have a fixed point (i.e., $f(x) > 0$ for all $x \in X$).
Fix $x_1 \in X$ and choose $x_2 \in Tx_1$ satisfying

$$d(x_1, x_2) < \frac{1}{\beta(d(x_1, x_2))} f(x_1) \tag{1}$$

Since $Tx_2 \neq \phi$, we can choose $x_3 \in Tx_2$ such that

$$f(x_2) \le d(x_2, x_3) \le \beta(d(x_1, x_2))d(x_1, x_2) \tag{2}$$

Also, as in equation (1), we have

$$d(x_2, x_3) < \frac{1}{\beta(d(x_2, x_3))} f(x_2) \tag{3}$$

From (2) and (3) we have

$$d(x_2, x_3) \leq \min\{\beta(d(x_1, x_2))d(x_1, x_2), \frac{1}{\beta(d(x_2, x_3))} f(x_2)\}$$

Thus

$$\beta(d(x_2, x_3))d(x_2, x_3) < f(x_2) \leq \beta(d(x_1, , x_2))d(x_1, x_2) < f(x_1)$$

Continuously, we can construct a sequence $\{x_n\} \in X$ such that $x_{n+1} \in Tx_n$ satisfying

$$\beta(d(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}) < f(x_{n+1}) \leq \beta(d(x_n, x_{n+1}))d(x_n, x_{n+1}) < f(x_n) \tag{4}$$

and

$$d(x_{n+1}, x_{n+2}) \leq \beta(d(x_{n+1}, x_n))d(x_{n+1}, x_n) \tag{5}$$

Since $\beta(t) < 1$ for all $t \in [0, \infty)$ we have $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$. From (4) and (5), the sequences $\{f(x_n)\}$ and $\{d(x_n, x_{n+1})\}$ are strictly decreasing.
Now we need to show that $\{x_n\}$ is a $(\sum, \neq)$- Cauchy sequence, thus we prove it in two steps:
Step 1 we need to show that all terms different. Suppose not i.e suppose $x_n = x_m$ for some $n > m$, where $n, m \in N$. Hence

$$\begin{aligned} f(x_m) &= \inf\{d(x_m, b) : b \in Tx_m\} \\ &= \inf\{d(x_n, b) : b \in Tx_n\} \\ &= f(x_n) \end{aligned}$$

which contradicts that $\{f(x_n)\}$ is strictly decreasing.
Step 2 We need to show that, $\sum d(x_n, x_{n+1}) < \infty$. Since $\{d(x_n, x_{n+1})\}$ is a non increasing sequence in R and bounded below, it converges to some non-negative real number (say δ). Also, we have $\lim_{s \to t^+} \sup \beta(s) < 1 \quad and \quad \beta(\delta) < 1$. Thus by lemma (2) there exist $r \in [0, 1) \quad and \quad \epsilon > 0$ such that $\beta(s) \leq r \quad \forall s \in [\delta, \delta + \epsilon)$. We can take $\mu \in N$ such that $\delta \leq d(x_n, x_{n+1}) \leq \delta + \epsilon \quad$ for all $n \in N$with $\quad n \geq \mu$. So,

$$\begin{aligned} \sum_{n=1}^{\infty} d(x_n, x_{n+1}) &\leq \sum_{n=1}^{\mu} d(x_n, x_{n+1}) + \sum_{n=\mu+1}^{\infty} d(x_n, x_{n+1}) \\ &\leq \sum_{n=1}^{\mu} d(x_n, x_{n+1}) + \sum_{n=1}^{\infty} r^n d(x_\mu, x_{\mu+1}) \\ &< \infty \end{aligned}$$

Thus $\{x_n\}$ is a $(\sum, \neq)$- Cauchy sequence in $(\sum, \neq)$ complete ν- generalize metric space and so it converges to some $y \in X$. By lemma (1); $\{x_n\}$ is a Cauchy sequence. From our assumption we choose $\{v_n\} \in Ty$ satisfy for any $n \in N$

$$d(x_{n+1}, v_n) \leq H(Tx_n, Ty) \leq \beta(d(x_n, y))d(x_n, y)$$

But $\{x_n\}$ converges to y, so $d(x_{n+1}, v_n) \to 0$ as $n \to \infty$. Thus we have $x_{n+1} \to y \quad and \quad x_{n+1} \to v_n$. Therefore, by lemma(1) $d(v_n, y) = 0$ as $n \to \infty$. So $d(Ty, y) = 0$ and $f(y) = 0$ which is a contradiction to our assumption. Thus, there exist $z \in X$ such that $f(z) = 0$ and this $z \in Tz$ is a fixed point. $\square$

Definition 6. *[14] T is called α- admissible if $\alpha(x,y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1$ where $\alpha : X \times X \to [0,\infty)$.*

Let Ψ be denote the family of all functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions;

- $\psi(s) = 0$ if and only if $s = 0$
- ψ is lower semicontinuous and non-decreasing.
- $\limsup_{s\to 0^+} \dfrac{s}{\psi(s)} < \infty$

The following lemma has been proved in [15] (see lemma 1.11), we prove it for multivalued map.

Lemma 4. *Let (X,d) be a ν- generalized metric space. Let T be a multivalued map on X. Let $\{x_n\}$ be a sequence in X defined by $x_0 \in X$ and $x_{n+1} \in Tx_n$ such that $x_n \neq x_{n+1}$. Assume that $\delta \in [0,1)$ such that*

$$d(x_n, x_{n+1}) \leq \delta d(x_{n-1}, x_n) \tag{6}$$

Then $x_n \neq x_m$ for all $n, m \in N$ distinct.

Proof. We will prove that $x_n \neq x_{n+k}$ for all $n \in N$ and $k \geq 1$. Suppose that is not true, i.e $x_n = x_{n+k}$ for some $n \in N$ and $k \geq 1$. By assumption, we have that $x_{n+1} = x_{n+k+1}$. Then from (6) we get;

$$d(x_n, x_{n+1}) = d(x_{n+k}, x_{n+k+1}) \leq \delta d(x_{n+k-1}, x_{n+k}) \leq ... \leq \delta^k d(x_n, x_{n+1}) < d(x_n, x_{n+1}) \tag{7}$$

which is contradiction. Thus we obtain $x_n \neq x_m$ for all $n \neq m$ in N. □

Theorem 5. *Let (X,d) be a $(\sum, \neq)$ complete ν- generalized metric space and let $T : X \to CB(X)$ be an α-admissible multivalued mapping satisfy the following:*

(i) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$
(ii) if $(y_n) \in Tx$ and (y_n) converge to y then $y \in Tx$.
(iii) $\alpha(x,y)H(Tx,Ty) \leq k(d(x,y))d(x,y)$ for any $x, y \in X$ and k is MT-function

Then T has a fixed point.

Proof. Define $\beta : [0,\infty) \to [0,1)$ as $\beta(t) = \dfrac{1+k(t)}{2}$ such that $\lim_{s\to t^+} \sup \beta(s) < 1$. Clearly $k(t) < \beta(t)$ for each $t \in [0,\infty)$. Let $x_0 \in X$ and choose $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. Assume $x_0 \neq x_1$ so, $\dfrac{1-k(d(x_0,x_1))}{2} d(x_0, x_1) > 0$. Since $Tx_1 \neq \phi$, choose $x_2 \in Tx_1$ such that;

$$\begin{aligned}
d(x_1, x_2) &\leq H(Tx_0, Tx_1) + \frac{1-k(d(x_0,x_1))}{2} d(x_0, x_1) \\
&\leq \alpha(x_0, x_1) H(Tx_0, Tx_1) + \frac{1-k(d(x_0,x_1))}{2} d(x_0, x_1) \\
&\leq k(d(x_0, x_1)) d(x_0, x_1) + \frac{1-k(d(x_0,x_1))}{2} d(x_0, x_1) \\
&\leq \beta(d(x_0, x_1)) d(x_0, x_1)
\end{aligned}$$

Since T is α- admissible, $x_1 \in Tx_0$ and $\alpha(x_0, x_1) \geq 1$ then $\alpha(Tx_0, Tx_1) \geq 1$ which implies $\alpha(x_1, x_2) \geq 1$. Similarly assume $x_1 \neq x_2$ we have $\frac{1 - k(d(x_1, x_2))}{2} d(x_1, x_2) > 0$ and choose $x_3 \in Tx_2$ such that;

$$
\begin{aligned}
d(x_2, x_3) &\leq H(Tx_1, Tx_2) + \frac{1 - k(d(x_1, x_2))}{2} d(x_1, x_2) \\
&\leq \alpha(x_0, x_1) H(Tx_0, Tx_1) + \frac{1 - k(d(x_0, x_1))}{2} d(x_0, x_1) \\
&\leq k(d(x_1, x_2)) d(x_1, x_2) + \frac{1 - k(d(x_1, x_2))}{2} d(x_1, x_2) \\
&\leq \beta(d(x_1, x_2)) d(x_1, x_2)
\end{aligned}
$$

Define $f(x) = \inf\{d(x, b) : b \in Tx\}$ and suppose that T does not have a fixed point (i.e $f(x) > 0$ for all $x \in X$). Fix $x_0 \in X$ and choose $x_1 \in Tx_0$ such that

$$
d(x_1, x_2) < \frac{1}{\beta(d(x_1, x_2))} f(x_1) \tag{8}
$$

Since $Tx_2 \neq \phi$, we can choose $x_3 \in Tx_2$ satisfying

$$
f(x_2) \leq d(x_2, x_3) \leq \beta(d(x_1, x_2)) d(x_1, x_2) \tag{9}
$$

Also, as in equation (8) we have

$$
d(x_2, x_3) < \frac{1}{\beta(d(x_2, x_3))} f(x_2) \tag{10}
$$

From (10) and (9) we have

$$
d(x_2, x_3) \leq \min\{\beta(d(x_1, x_2)) d(x_1, x_2), \frac{1}{\beta(d(x_2, x_3))} f(x_2)\}
$$

Thus

$$
\beta(d(x_2, x_3)) d(x_2, x_3) < f(x_2) \leq \beta(d(x_1, , x_2)) d(x_1, x_2) < f(x_1)
$$

In this way, we can construct a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ and $\alpha(x_n, x_{n+1}) \geq 1$ satisfying

$$
\beta(d(x_{n+1}, x_{n+2}) d(x_{n+1}, x_{n+2}) < f(x_{n+1}) \leq \beta(d(x_n, x_{n+1})) d(x_n, x_{n+1}) < f(x_n) \tag{11}
$$

and

$$
d(x_{n+1}, x_{n+2}) \leq \beta(d(x_{n+1}, x_n)) d(x_{n+1}, x_n) \tag{12}
$$

Since $\beta(t) < 1$ for all $t \in [0, \infty)$ we have $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$. From (11) and (12), the sequences $\{f(x_n)\}$ and $\{d(x_n, x_{n+1})\}$ are strictly decreasing. Thus x_n's are all different. Since $\beta(t) < 1$ for all $t \in [0, \infty)$ and $\{d(x_n, x_{n+1})\}$ is a non increasing sequence in R, hence it converges to some non-negative real number (say δ). Also, we have $\lim_{s \to t^+} \sup \beta(s) < 1 \quad and \quad \beta(\delta) < 1$, then there exist

$r \in [0,1)$and $\epsilon > 0$ such that $\beta(s) \leq r \quad \forall s \in [\delta, \delta+\epsilon)$. We can take $\mu \in N$ such that $\delta \leq d(x_n, x_{n+1}) \leq \delta + \epsilon \quad$ for all $n \in N \quad with \quad n \geq \mu$. So,

$$\begin{aligned}\sum_{n=1}^{\infty} d(x_n, x_{n+1}) &\leq \sum_{n=1}^{\mu} d(x_n, x_{n+1}) + \sum_{n=\mu+1}^{\infty} d(x_n, x_{n+1}) \\ &\leq \sum_{n=1}^{\mu} d(x_n, x_{n+1}) + \sum_{n=1}^{\infty} r^n d(x_\mu, x_{\mu+1}) \\ &< \infty\end{aligned}$$

Thus $\{x_n\}$ is a $(\sum, \neq)$- Cauchy sequence in $(\sum, \neq)$ complete ν- generalize metric space and hence it converges to some $y \in X$. By lemma(1), $\{x_n\}$ is a Cauchy sequence. From our assumption we choose $\{v_n\} \in Ty$ such that

$$d(x_{n+1}, v_n) \leq H(Tx_n, Ty) \leq \beta(d(x_n, y))d(x_n, y)$$

for any $n \in N$. But $\{x_n\}$ converges to y, so $d(x_{n+1}, v_n) \to 0$ as $n \to \infty$. Thus we have $x_{n+1} \to y \quad and \quad x_{n+1} \to v_n$.
Therefore by lemma 1 part (2) $d(v_n, y) = 0$ as $n \to \infty$. Thus $d(Ty, y) = 0$ and $f(y) = 0$ which is a contradiction to our assumption. So there exist $z \in X$ such that $f(z) = 0$ and this $z \in Tz$ is a fixed point. $\square$

Theorem 6. *Let (X, d) be a $(\sum, \neq)$ complete ν- generalized metric space and let $T : X \to CB(X)$ be a multi valued map satisfying:*

$$\psi(H(Tx, Ty)) \leq \alpha(\psi(d(x, y)))\psi(d(x, y)) \quad \text{for each} \quad x, y \in X$$

where $\alpha : [0, \infty) \to [0, 1)$ satisfies $\limsup_{s \to t^+} \alpha(s) < 1$ for each $t \in [0, \infty)$ and $\psi \in \Psi$. Then T has a fixed point.

Proof. Let $\beta : [0, \infty) \to [0, 1)$ defined by $\beta(t) = \dfrac{1 + \alpha(t)}{2}$. It is not difficult to show that $\lim_{s \to t^+} \sup \beta(s) < 1$. Since ψ is an increasing function, then we have:

$$\begin{aligned}&\max\left\{\sup_{u \in Tx} \psi(d(u, Ty)), \sup_{v \in Ty} \psi(d(v, Tx))\right\} \\ &= \max\left\{\psi(\sup_{u \in Tx} d(u, Ty)), \psi(\sup_{v \in Ty} d(v, Tx))\right\} \\ &= \psi\left(H(Tx, Ty)\right) \leq \beta(\psi(d(x, y)))\psi(d(x, y))\end{aligned} \tag{13}$$

For each $x \in X$ and $y \in Tx$ there exist an element $z \in Ty$ such that

$$\psi(d(y, z)) \leq \beta(\psi(d(x, y)))\psi(d(x, y))$$

Thus we can construct a sequence $\{x_n\} \in X$ defined as $x_{n+1} \in Tx_n$ and satisfying

$$\psi(d(x_{n+1}, x_{n+2}) \leq \beta(\psi(d(x_n, x_{n+1}))\psi(d(x_n, x_{n+1})); \quad \text{for each} \quad n \in N \tag{14}$$

Since $\beta(t) < 1$ for any $t \in [0, \infty)$ so from (14) we have

$$\psi(d(x_{n+1}, x_{n+2}) < \psi(d(x_n, x_{n+1})); \quad \text{for each} \quad n \in N \tag{15}$$

Also, ψ is an increasing function and that leds to $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$. Thus the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and hence by lemma (4) we get x_n's all different. Now, we need to show that $\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty$. We have decreasing sequence $\{d(x_n, x_{n+1})\}$ and bounded below, hence it is converges to some non-negative real number (say τ). We have also $\psi(\tau) \leq \psi(d(x_n, x_{n+1}))$ for each $n \in N$. Thus,

$$\psi(\tau) \leq \lim_{n\to\infty} \psi(d(x_n, x_{n+1})) = \delta = 0$$

Since $\psi(s) = 0 \quad \text{if and only if} \quad s = 0$ then $\tau = 0$. By lemma (2), there exist $r \in [0, 1)$ such that , we can take $\mu \in N$ such that $\delta \leq \psi(d(x_n, x_{n+1})) < \delta + \epsilon$.

$$\begin{aligned}
\sum_{n=1}^{\infty} \psi(d(x_n, x_{n+1})) &\leq \sum_{n=1}^{\mu} \psi(d(x_n, x_{n+1})) + \sum_{n=\mu+1}^{\infty} \psi(d(x_n, x_{n+1})) \\
&\leq \sum_{n=1}^{\mu} \psi(d(x_n, x_{n+1})) + \sum_{n=1}^{\infty} r^n \psi(d(x_\mu, x_{\mu+1})) \\
&< \infty
\end{aligned}$$

By defintion of ψ, we have,

$$\lim_{n\to\infty} \sup \frac{d(x_n, x_{n+1})}{\psi(d(x_n, x_{n+1}))} \leq \lim_{s\to 0^+} \frac{s}{\psi(s)} < \infty$$

Hence the sequence $\{x_n\} is (\sum, \neq)$- Cauchy sequence in complete $(\sum, \neq)$ generalized metric space. By lemma (1), it is Cauchy and then it is converge to some $z \in X$. Since ψ is lower semi continuous and non-decreasing, then

$$\begin{aligned}
\psi(d(z, Tz)) &\leq \liminf \psi(d(x_{n+1}, Tz) \leq \liminf \psi(H(Tx_n, Tz)) \\
&\leq \liminf \beta(\psi(d(x_n, z)))\psi(d(x_n, z)) \leq \liminf \psi(d(x_n, z)) \\
&= \lim_{s\to 0^+} \psi(s) = \lim \psi(d(x_n, x_{n+1})) = 0
\end{aligned}$$

Therefore $\psi(d(z, Tz)) = 0$. Thus by the definition of ψ and since Tz closed we have $z \in Tz$ is a fixed point. □

4. Discussion

In this paper we extended Mizoguchi- Takahashi theorem from metric space to ν- generalized metric spaces and proved some of its generalization (theorem 5 and 6). Thus, one can take this space as a research point to study it deeply and discover more a bout its properties. Also, many fixed point theorem which proved in metric space and does not yet prove in the new one is still as open problem.

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